Solution to Assignment 6

Section 7.2

18. If $f \equiv 0$, then result is trivial. Otherwise, since f is continuous on $[a,b]$, there exists $x_0 \in [a,b]$ s.t. $\sup f = f(x_0) > 0$. By continuity, for each small $\varepsilon > 0$, there is some δ such that $|f(x) - f(x_0)| < \varepsilon$, for all $x \in [x_0 - \delta, x_0 + \delta] \cap [a,b]$. Hence

$$\delta(f(x_0) - \varepsilon)^n < \int_{(x_0-\delta,\, x_0+\delta)\cap[a,b]} f^n \le \int_a^b f^n \le \int_a^b f(x_0)^n = f(x_0)^n(b-a)$$

$$\delta^{1/n}(f(x_0) - \varepsilon) < M_n = \left(\int_a^b f^n\right)^{1/n} \le f(x_0)(b-a)^{1/n}$$

Note that $\lim_{n\to\infty} a^{1/n} = 1 \ \forall\ a > 0$. Letting $n \to \infty$, by the squeeze theorem,

$$f(x_0) - \varepsilon \le \liminf_{n\to\infty} M_n \le \limsup_{n\to\infty} M_n \le f(x_0)$$

Letting $\varepsilon \to 0$, $\lim_{n\to\infty} M_n = f(x_0) = \sup\{f(x) : x \in [a,b]\}$.

19. Let P_n be the equal length partition of $[-a,0], -a = x_0 < x_1 < \cdots < x_n = 0$, where $x_j = -a + ja/n,\ j = 0, \cdots, n$. Then we have

$$\int_{-a}^0 f = \lim_{n\to\infty} \sum_j f(x_j)\frac{a}{n} \ ,$$

see Theorem 2.6. On the other hand, $-x_j, j = 0, \cdots, n$, becomes a partition Q_n on $[0,a]$. Therefore,

$$\int_0^a f = \lim_{n\to\infty} \sum_j f(-x_j)\frac{a}{n} \ .$$

Using $f(-x) = f(x)$, we see that

$$\sum_j f(-x_j)\frac{a}{n} = \sum_j f(x_j)\frac{a}{n} \ ,$$

hence

$$\int_{-a}^0 f = \int_0^a f \ .$$

When f is odd, follow the same line but now using $\sum_j f(-x_j)\dfrac{a}{n} = -\sum_j f(x_j)\dfrac{a}{n}$ to get

$$\int_{-a}^0 f = -\int_0^a f \ .$$

Section 7.3

10. We let $F(t) = \int_a^t f$. Then $G(x) = \int_a^{\nu(x)} f = F(\nu(x))$. Applying the Chain Rule and then the Second Fundamental Theorem,

$$G'(x) = F'(\nu(x))\nu'(x) = f(\nu(x))\nu'(x) \ .$$

11. First,

$$F(x) = \int_0^{x^2} \frac{1}{1+t^3} dt \ .$$

By taking $\nu(x) = x^2$ and applying the previous problem, we have

$$F'(x) = \frac{1}{1+x^6} \times 2x = \frac{2x}{1+x^6} \ .$$

Next, write

$$F(x) = \int_0^x \sqrt{1+t^2} dt - \int_0^{x^2} \sqrt{1+t^2} dt \ ,$$

and apply the previous problem separately to get

$$F'(x) = \sqrt{1+x^2} - 2x\sqrt{1+x^4} \ .$$

16. Differentiate both sides to get

$$f(x) = -f(x) \ ,$$

after noting

$$\int_x^1 f = \int_0^1 f - \int_0^x f \ .$$

Check the assumption for the Second Fundamental Theorem.

Supplementary Exercise

1. Order the rational numbers in $[0,1]$ into a sequence $\{z_n\}$ and define

$$\varphi(x) = \sum_{\{j,\ z_j < x\}} \frac{1}{2^j} \ .$$

Show that φ is continuous at every irrational number but discontinuous at every rational number in $(0,1)$.

Solution. Let x be rational. Then $x = z_k$ for some k. From the definition of ϕ we immediately obtain $\phi(z_k^+) - \phi(z_k^-) = 1/2^k$, so it has a jump at z_k. On the other hand, for $\varepsilon > 0$, we fix a large j_0 such that $\sum_{j=j_0}^\infty 2^{-j} < \varepsilon$. The finite points $z_1, \cdots, z_{j_0}$ are disjoint from x and we can find some $\delta > 0$ such that $(x-\delta, x+\delta)$ does not contain any $z_1, \cdots, z_{j_0}$. That is, $z_j \in (x-\delta, x+\delta)$ implies $j > j_0$. It follows that for $y \in (x-\delta, x+\delta), y > x$,

$$0 < \phi(y) - \phi(x) = \sum_{\{j:\ x \le z_j < y\}} \frac{1}{2^j} \le \sum_{j=j_0+1}^\infty \frac{1}{2^j} = \frac{1}{2^{j_0}} < \varepsilon \ .$$

Similarly, we have $0 < \phi(x) - \phi(y) < \varepsilon$ for $y \in [x-\delta, x)$.

Note. This function is strictly increasing. Since monotone functions are integrable, this example shows how complicated an integrable function could be. It has countably many discontinuity points spreading densely over the interval. Thomae's function is another example of the same nature, although it is not monotone.

2. Give two integrable functions f and Φ so that $\Phi \circ f$ is not integrable. Hint: Take f to be the Thomae's function.

   Solution. Take f to be the Thomae's function which has been shown to be integrable on $[0,1]$. Next consider $\Phi(x) = 0$ if $x = 0$ and $\Phi(x) = 1$ otherwise. Φ is bounded and has only one discontinuity point at $x = 0$ and hence integrable. However, the composite function $\Phi \circ f$ satisfies $\Phi \circ f(x) = 1, x \in \mathbb{Q}$ and $\Phi \circ f(x) = 0$ otherwise. It is not integrable, see Example 2.2.

3. Evaluate the following limits:

   (a)
$$\lim_{n\to\infty} \Big(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\Big);$$

   (b)
$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}.$$

   Solution. (a) We observe
$$\frac{1}{n+1} + \cdots + \frac{n}{n+n} = \frac{1}{n}\sum_{j=1}^{n} \frac{n}{n+j} = \sum_{j=1}^{n} \frac{1}{1+j/n} .$$

   Using the integrability of the function $f(x) = 1/(1+x)$, we see that
$$\lim_{n\to\infty} \Big(\frac{1}{n+1} + \cdots + \frac{n}{n+n}\Big) = \int_0^1 \frac{1}{1+x}dx = \log(1+x)\Big|_0^1 = \log 2 .$$

   (b) Taking log,
$$\frac{1}{n}\log n! - \log n = \frac{1}{n}\sum_{j=1}^{n} \big(\log j - \log n\big) = \frac{1}{n}\sum_{j=1}^{n} \log\frac{j}{n} .$$

   Letting $g(x) = \log x,\ x \in [0,1]$,
$$\lim_{n\to\infty} \log\frac{(n!)^{1/n}}{n} = \lim_{n\to\infty} \frac{1}{n}\log n! - \log n = \lim_{j\to\infty} \frac{1}{n}\sum_{n=1}^{n} \log\frac{j}{n} = \int_0^1 \log x dx = -1 .$$

   Hence
$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = e^{-1} .$$

4. Evaluate the following integrals
$$\int_0^a x^2\sqrt{a^2 - x^2}dx .$$

   Solution. WLOG take $a > 0$. Use the change of variables $x = a\sin\theta,\ \theta \in [0, \pi/2]$. Then

$dx/d\theta = a\cos\theta$ on $[0, \pi/2]$.

$$\begin{aligned}
\int_0^a x^2\sqrt{a^2-x^2}\ dx &= \int_0^{\pi/2} a^2\sin^2\theta(|\ a\ |\cos\theta)(a\cos\theta)\ d\theta \\
&= a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\ d\theta \\
&= \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\ d\theta \\
&= \frac{a^4}{4}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)\ d\theta \\
&= \frac{a^4}{8}\left(\theta-\frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/2} \\
&= \frac{\pi a^4}{16}\ .
\end{aligned}$$

5. Prove the following formula: For any "nice" function f

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

Solution.

$$\begin{aligned}
\int_0^\pi xf(\sin x)\ dx &= \int_0^{\pi/2} xf(\sin x)\ dx + \int_{\pi/2}^\pi xf(\sin x)\ dx \\
&= \int_0^{\pi/2} xf(\sin x)\ dx + \int_{\pi/2}^0 (\pi-u)f(\sin(\pi-u))(-1)\ du \\
&= \int_0^{\pi/2} xf(\sin x)\ dx + \int_0^{\pi/2}(\pi-x)f(\sin x)\ dx = \pi\int_0^{\pi/2} f(\sin x)\ dx\ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^\pi f(\sin x)\ dx &= \int_0^{\pi/2} f(\sin x)\ dx + \int_{\pi/2}^0 f(\sin(\pi-u))(-1)\ du \\
&= 2\int_0^{\pi/2} f(\sin x)\ dx\ .
\end{aligned}$$

Hence,

$$\int_0^\pi xf(\sin x)\ dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\ dx\ .$$

6. Evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.$$

Hint: Use the previous problem.

Solution.

$$\begin{aligned}
\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx &= \int_0^\pi \frac{x\sin x}{2-\sin^2 x}\,dx \\
&= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{2-\sin^2 x}\,dx \\
&= -\frac{\pi}{2}\int_0^\pi \frac{-\sin x}{1+\cos^2 x}\,dx \\
&= \frac{-\frac{\pi}{2}\int_0^\pi d(\cos x)}{1+\cos^2 x} \\
&= -\frac{\pi}{2}\mathrm{Arctan}\cos x\Big|_0^\pi \\
&= \frac{\pi^2}{4}\ .
\end{aligned}$$

7. For a continuous function f on $[-a,a]$, prove that when it satisfies

$$\int_{-a}^{a} fg = 0,$$

for all even, integrable functions g, it must be an odd function.

Solution. Step 1. Define:

$$f = f_e + f_o$$
$$f_e = \frac{f(x)+f(-x)}{2}$$
$$f_o = \frac{f(x)-f(-x)}{2}$$

Note f_e is even while f_o is odd.
Then,

$$0 = \int_{-a}^{a} fg = \int_{-a}^{a} f_e g + \int_{-a}^{a} f_o g\ .$$

Use change of variables,

$$\begin{aligned}
\int_{-a}^{0} f_o g &= \int_{-a}^{0} f_o(x)g(x)dx \\
&= \int_{-a}^{0} f_o(-x)g(-x)dx \\
&= \int_{0}^{a} f_o(-x)g(x)dx \\
&= -\int_{0}^{a} f_o(x)g(x)dx\ .
\end{aligned}$$

Therefore, $\int_{-a}^{a} f_o g = 0$.
It follows that

$$0 = \int_{-a}^{a} f_e g\ .$$

As f_e is even, set $g = f_e$, $\int_{-a}^{a} {f_e}^2 = 0 \Rightarrow f_e \equiv 0$, so $f = f_o$ is odd. At the last we use the continuity of f (so are f_e and f_o).

8. Evaluate the following integrals:

   (a)
$$\int_0^{\pi} x \sin x dx \ ,$$

   (b)
$$\int_0^1 \text{Arccos} x dx.$$

   The inverse cosine function Arccos maps $[-1,1]$ to $[0,\pi]$.

   Solution. (a)
$$\int_0^{\pi} x \sin x \ dx = \int_0^{\pi} x \ (-\cos x)' dx = (-x\cos x)\Big|_0^{\pi} + \int_0^{\pi} \cos x \ dx = \pi \ .$$

   (b) Let $x = \cos\theta, \ \theta \in [0,\pi/2]$. Then $dx/d\theta = -\sin\theta$ on $[0,\pi/2]$. Then
$$\begin{aligned}
\int_0^1 \text{Arccos } x \ dx &= \int_{\pi/2}^0 \theta(-\sin\theta)\, d\theta \\
&= (-\theta\cos\theta)\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \ d\theta \\
&= 1 \ .
\end{aligned}$$

9. Evaluate the following integrals:

   (a)
$$\int_0^1 (1-x^2)^n dx \ ,$$

   (b)
$$\int_0^1 x^m (\log x)^n dx, \quad m,n \in \mathbb{N}.$$

   Solution. (a)
   Let $x = \sin\theta, \ \theta \in [0,\pi/2]$. Then $dx/d\theta = \cos\theta$ on $[0,\pi/2]$.
$$\begin{aligned}
I_n \equiv \int_0^1 (1-x^2)^n \ dx &= \int_0^{\pi/2} \cos^{2n+1}\theta \ d\theta \\
&= \int_0^{\pi/2} \cos^{2n}\theta \ (\sin\theta)' \ d\theta \\
&= (\cos^{2n}\theta\sin\theta)\Big|_0^{\pi/2} + 2n\int_0^{\pi/2} \cos^{2n-1}\theta \sin^2\theta \ d\theta \\
&= 2n \int_0^{\pi/2} \cos^{2n-1}\theta(1-\cos^2\theta) \ d\theta \\
&= 2n \int_0^{\pi/2} \cos^{2n-1}\theta \ d\theta - 2n \int_0^{\pi/2} \cos^{2n+1}\theta \ d\theta = 2nI_{n-1} - 2nI_n \ .
\end{aligned}$$

Therefore,

$$\begin{aligned} I_n &= \frac{2n}{2n+1} I_{n-1} \\ &= \frac{2n}{2n+1} \frac{2n-2}{2n-1} \cdots \frac{2}{3}\ I_0 \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!} \int_0^{\pi/2} \cos\theta \ d\theta \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!} \sin\theta \Big|_0^{\pi/2} \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!}\ . \end{aligned}$$

(b)

$$\begin{aligned} I_{m,n} \equiv \int_0^1 x^m (\log x)^n \ dx &= \frac{1}{m+1} \lim_{\varepsilon \to 0} \int_\varepsilon^1 (\log x)^n (x^{m+1})' \, dx \\ &= \lim_{\varepsilon \to 0} \frac{x^{m+1}}{m+1} (\log x)^n \Big|_\varepsilon^1 - \frac{1}{m+1} \int_0^1 x^{m+1} (n (\log x)^{n-1}) \frac{1}{x} \ dx \\ &= -\frac{n}{m+1} \int_0^1 x^m (\ln x)^{n-1} \ dx \\ &= -\frac{n}{m+1} I_{m,n-1} \\ &= (-1)^n \frac{n!}{(m+1)^n} I_{m,0} \\ &= (-1)^n \frac{n!}{(m+1)^n} \int_0^1 x^m \ dx \\ &= \frac{(-1)^n n!}{(m+1)^n} \frac{x^{m+1}}{m+1} \Big|_0^1 \\ &= \frac{(-1)^n n!}{(m+1)^{n+1}}\ . \end{aligned}$$